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SYMMETRY OPERATORS AND GENERAL SOLUTIONS OF THE EQUATIONS OF THE LINEAR THEORY OF ELASTICITY

N. I. Ostrosablin

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Different variants of general solutions, i.e., representations of stresses or displacements in terms of arbitrary independent functions (for example, harmonic and biharmonic) such that the equations of equilibrium or motion are satisfied identically, are known in the theory of elasticity [1–13]. The general solutions of Kelvin–Lamé [13], Galerkin [6], and Papkovich–Neuber [1, 3, 4] are most often used. The problem of generality and completeness of the solutions has been discussed in many works (see, for example, [2–5, 8, 9, 14–24]).

In the present work, using the equations of the linear theory of elasticity as an example, we show that for each general solution there is a formula for obtaining new solutions, i.e., some symmetry operator [25]. For an isotropic material, the symmetry operators are found for the solutions by Kelvin–Lamé, Galerkin, and Papkovich–Neuber and the generality of these solutions is proved. Some other symmetry operators are presented in [26, 27].

Let us consider linear differential operators of the form

$$A_{ij} = a_{ij}(x_s) + a_{ijk}(x_s)\partial_k + a_{ij(kl)}(x_s)\partial_{kl} + a_{ij(klm)}(x_s)\partial_{klm} + \ldots \tag{1}$$

and formally conjugate operators

$$A^*_{ji} = a_{ij}(x_s) - \partial_k a_{ijk}(x_s) + \partial_{kl}a_{ij(kl)}(x_s) - \partial_{klm}a_{ij(klm)}(x_s) + \ldots . \tag{2}$$

Here x_s are independent variables; ∂_k is the derivative with respect to the variable x_k; repeated subscripts indicate summation, and subscripts in parentheses denote a symmetric function of these subscripts.

Let us assume that $A^* = A$, $D^* = D$, and $AC = BD$; then $C^*A = DB^*$. For given A and B one can always find [28] operators C and D satisfying these relations.

If $u = C\varphi$, where $D\varphi = 0$, then the equation

$$Au = AC\varphi = BD\varphi = 0 \tag{3}$$

holds. If $\varphi = B^*\tilde{u}$, where $A\tilde{u} = 0$, then the equation

$$D\varphi = DB^*\tilde{u} = C^*A\tilde{u} = 0 \tag{4}$$

is valid. Thus, according to the formulas

$$u = C\varphi, \qquad \varphi = B^*\tilde{u} \tag{5}$$

solutions of Eqs. (3) and (4) transform into one another. If $A\tilde{u} = 0$, then it follows from (3)–(5) that $u = CB^*\tilde{u}$ is a new solution:

$$Au = ACB^*\tilde{u} = BDB^*\tilde{u} = BC^*A\tilde{u} = 0. \tag{6}$$

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For the linear equations $Au = 0$ the operator Q is called a symmetry operator [29] if $AQ - QA = RA$. The symmetry operator transfers the solution of the equation $A\tilde{u} = 0$ into a new solution $u = Q\tilde{u}$: $Au = AQ\tilde{u} = (Q + R)A\tilde{u} = 0$. Hence and from (6) it is obvious that $Q = CB^*$ is a symmetry operator and $R = BC^* - CB^*$.

The general solution of Eq. (3) has the form

$$u = C\varphi, \qquad D\varphi = f, \qquad Bf = 0, \tag{7}$$

where $f \in \mathrm{Ker}B = \{f. Bf = 0\}$. If the operators are such that $D\,\mathrm{Ker}\,C = \mathrm{Ker}B$, then the general solution of Eq. (3) is [28]

$$u = C\varphi, \qquad D\varphi = 0. \tag{8}$$

Actually, there is ψ such that $u = C\psi$, $D\psi = f$, $Bf = 0$. Since $D\,\mathrm{Ker}\,C = \mathrm{Ker}B$, there is $g \in \mathrm{Ker}\,C$ such that $f = Dg$. Then $D(\psi - g) = 0$, $u = C(\psi - g)$. Denoting $\varphi = \psi - g$, we obtain (8).

Thus, the general solutions of the equation $Au = 0$ are based on the relation $AC = BD$ and are given by formulas (7) or (8). The formula of producing new solutions $u = CB^*\tilde{u}$ or the symmetry operator $Q = CB^*$ correspond to each general solution. The approach cited makes it possible to find symmetry operators of the form $Q = CB^*$, and if a symmetry operator Q is known which can be presented in this form one can find [28] the operator D and obtain the general solutions (7) or (8).

The general solutions known in the theory of elasticity are usually written in the form (8) without checking the validity of the condition $D\,\mathrm{Ker}\,C = \mathrm{Ker}B$. But if this condition is not fulfilled, then (8) will not be a general or a complete solution, since in this case the solutions corresponding to the nonhomogeneous equation $D\varphi = f \in \mathrm{Ker}B$ are lost. We further check the condition $D\,\mathrm{Ker}\,C = \mathrm{Ker}B$ and find the symmetry operators $Q = CB^*$ for the case of an isotropic material for the solutions by Kelvin–Lamé, Galerkin, and Papkovich–Neuber.

In the classical Kelvin–Lamé solution [13] for the operator

$$A_{ij} = A_{ji} = (\lambda + \mu)\partial_{ij} + (\mu\partial_{kk} - \rho\partial_{..})\delta_{ij} = A^*_{ji} \tag{9}$$

(λ and μ are Lamé's constants, ρ is the constant density of the material, δ_{ij} is the Kronecker symbol, $\partial.$ is the time derivative) the displacements u_j are represented as:

$$u_j = \partial_j\varphi + \varepsilon_{jps}\partial_p\psi_s. \quad \partial_i\psi_i = 0, \quad [(\lambda + 2\mu)\partial_{kk} - \rho\partial_{..}]\varphi = 0, \quad (\mu\partial_{kk} - \rho\partial_{..})\psi_s = 0, \quad s = 1,2,3 \tag{10}$$

(ε_{jps} are the Levi-Civita symbols). From (10), we obtain

$$C = \begin{bmatrix} \partial_1 & 0 & -\partial_3 & \partial_2 \\ \partial_2 & \partial_3 & 0 & -\partial_1 \\ \partial_3 & -\partial_2 & \partial_1 & 0 \end{bmatrix}, \qquad C^* = -\begin{bmatrix} \partial_1 & \partial_2 & \partial_3 \\ 0 & \partial_3 & -\partial_2 \\ -\partial_3 & 0 & \partial_1 \\ \partial_2 & -\partial_1 & 0 \end{bmatrix} = -C' \tag{11}$$

(primes denote transposition of the matrix). To find AC we substitute u_j from (10) into (9):

$$A_{ij}u_j = [(\lambda+\mu)\partial_{ij} + (\mu\partial_{kk} - \rho\partial_{..})\delta_{ij}]\,(\partial_j\varphi + \varepsilon_{jps}\partial_p\psi_s) = [(\lambda+2\mu)\partial_{kk} - \rho\partial_{..}]\,\partial_i\varphi + (\mu\partial_{kk} - \rho\partial_{..})\varepsilon_{ips}\partial_p\psi_s. \tag{12}$$

It is obvious from (12) that

$$AC = \begin{bmatrix} \partial_1 & 0 & -\partial_3 & \partial_2 \\ \partial_2 & \partial_3 & 0 & -\partial_1 \\ \partial_3 & -\partial_2 & \partial_1 & 0 \end{bmatrix} \begin{bmatrix} (\lambda + 2\mu)\partial_{kk} - \rho\partial_{..} & 0 & 0 & 0 \\ 0 & \mu\partial_{kk} - \rho\partial_{..} & 0 & 0 \\ 0 & 0 & \mu\partial_{kk} - \rho\partial_{..} & 0 \\ 0 & 0 & 0 & \mu\partial_{kk} - \rho\partial_{..} \end{bmatrix} = CD = BD,$$

i.e., $B = C$ and $B^* = C^* = -C'$. and D is the diagonal matrix. From the relation $C^*A = DB^*$ it follows that $-C'A = D(-C')$, $C'A = DC'$. Then $\varphi = C'\tilde{u}$ and $D\varphi = DC'\tilde{u} = C'A\tilde{u} = 0$.

Now taking into account (11), we write φ and ψ_j in terms of $\tilde{u}$:

$$\varphi = \partial_i\tilde{u}_i, \qquad \psi_j = -\varepsilon_{jps}\partial_p\tilde{u}_s. \tag{13}$$

Since $u = C\varphi$ and $\varphi = C'\widetilde{u}$, $u = CC'\widetilde{u}$ is a new solution: $Au = ACC'\widetilde{u} = CDC'\widetilde{u} = CC'A\widetilde{u} = 0$. From (10) and (13), we obtain the formula for deriving solutions

$$u_j = \partial_j\partial_i\widetilde{u}_i + \varepsilon_{jps}\partial_p(-\varepsilon_{smn}\partial_m\widetilde{u}_n) = \delta_{jn}\partial_{pp}\widetilde{u}_n = \partial_{pp}\widetilde{u}_j, \tag{14}$$

where $Q = CC' = \delta_{jn}\partial_{pp}$ is the symmetry operator and $\widetilde{u}_j$, the solution of the equation

$$[(\lambda+\mu)\partial_{ij} + (\mu\partial_{kk} - \rho\partial_{..})\delta_{ij}]\,\widetilde{u}_j = 0. \tag{15}$$

It follows from (13) that the second equation (10) is always valid:

$$\partial_j\psi_j = -\partial_j\varepsilon_{jps}\partial_p\widetilde{u}_s = -\varepsilon_{jps}\partial_{jp}\widetilde{u}_s \equiv 0.$$

The kernel of the operator $C = B$ is determined from the equations

$$Cg = \begin{bmatrix} \partial_1 g + \partial_2 g_3 - \partial_3 g_2 \\ \partial_2 g + \partial_3 g_1 - \partial_1 g_3 \\ \partial_3 g + \partial_1 g_2 - \partial_2 g_1 \end{bmatrix} = 0,$$

whose solution, as can be easily verified, is as follows:

$$g = \partial_i f_i, \qquad \partial_{kk} f_i = 0, \qquad g_s = \partial_s f - \varepsilon_{smn}\partial_m f_n \tag{16}$$

(f is an arbitrary function). We now find $p = Dg$:

$$p = [(\lambda+2\mu)\partial_{kk} - \rho\partial_{..}]\partial_i f_i = \partial_i[(\lambda+2\mu)\partial_{kk} - \rho\partial_{..}]f_i = \partial_i(-\rho\partial_{..}f_i),$$

$$p_s = (\mu\partial_{kk} - \rho\partial_{..})(\partial_s f - \varepsilon_{smn}\partial_m f_n)$$

$$= \partial_s[(\mu\partial_{kk} - \rho\partial_{..})f] - \varepsilon_{smn}\partial_m[(\mu\partial_{kk} - \rho\partial_{..})f_n] = \partial_s[(\mu\partial_{kk} - \rho\partial_{..})f] - \varepsilon_{smn}\partial_m(-\rho\partial_{..}f_n).$$

Denoting here $q = (\mu\partial_{kk} - \rho\partial_{..})f$, $q_i = -\rho\partial_{..}f_i$, we find that $D\,\mathrm{Ker}\,C$ has the form (16)

$$p = \partial_i q_i, \qquad \partial_{kk} q_i = 0, \qquad p_s = \partial_s q - \varepsilon_{smn}\partial_m q_n.$$

This means that $D\,\mathrm{Ker}\,C = \mathrm{Ker}\,B = \mathrm{Ker}\,C$, the Kelvin–Lamé solution is complete, and it suffices to write it in the form (10). Also, the formulas (13)–(15) hold true.

It should be noted that in statics, as is seen from (10), the functions φ and ψ_s are harmonic, and then $\partial_j u_j = \partial_{jj}\varphi = \partial_{jj}\partial_i\widetilde{u}_i = 0$, i.e., there is no change of volume. This is related to the fact that following formula (14) an arbitrary solution $\widetilde{u}_j$ is transferred into a solution u_j, whose volume remains unchanged.

Let us now consider the Galerkin solution [6], which we write as [10]

$$\begin{aligned} u_j = C_{jk}\varphi_k &= [-(\lambda+\mu)\partial_{jk} + ((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{jk}]\varphi_k, \\ D_{jk}\varphi_k &= (\mu\partial_{pp} - \rho\partial_{..})((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{jk}\varphi_k = 0. \end{aligned} \tag{17}$$

It is obvious that $C^*_{kj} = C_{jk}$ in (17). Then we find

$$\begin{aligned} A_{ij}C_{jk} &= [(\lambda+\mu)\partial_{ij} + (\mu\partial_{pp} - \rho\partial_{..})\delta_{ij}][-(\lambda+\mu)\partial_{jk} + ((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{jk}] \\ &= (\mu\partial_{pp} - \rho\partial_{..})((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{ij}\delta_{jk} = B_{ij}D_{jk}. \end{aligned} \tag{18}$$

It follows from (18) that the following variants are possible:

1) $\quad B_{ij} = \delta_{ij}, \qquad D_{jk} = (\mu\partial_{pp} - \rho\partial_{..})((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{jk};$

2) $\quad B_{ij} = (\mu\partial_{pp} - \rho\partial_{..})\delta_{ij}, \qquad D_{jk} = ((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{jk};$

3) $\quad B_{ij} = ((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{ij}, \qquad D_{jk} = (\mu\partial_{pp} - \rho\partial_{..})\delta_{jk};$

4) $\quad B_{ij} = (\mu\partial_{pp} - \rho\partial_{..})((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{ij}, \qquad D_{jk} = \delta_{jk}.$

Variants 1 and 4 are equipotent and to them correspond the solution (17). In addition, bearing the foregoing in mind, we obtain

$$\varphi_j = B^*_{ji}\widetilde{u}_i = B_{ij}\widetilde{u}_i = \delta_{ij}\widetilde{u}_i = \widetilde{u}_j, \quad A_{ij}\widetilde{u}_j = [(\lambda+\mu)\partial_{ij} + (\mu\partial_{kk} - \rho\partial_{..})\delta_{ij}]\widetilde{u}_j = 0; \tag{19}$$

$$u_j = C_{jk}B^*_{ki}\tilde{u}_i = C_{jk}B_{ik}\tilde{u}_i = C_{jk}\delta_{ik}\tilde{u}_i = C_{jk}\tilde{u}_k = [-(\lambda+\mu)\partial_{jk} + ((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{jk}]\tilde{u}_k. \quad (20)$$

Formulas (17) are the ordinary Galerkin solution, (19) is the expression of φ_j in terms of $\tilde{u}_j$, and (20) is the formula for obtaining new solutions (C_{jk} is the symmetry operator).

For variant 2, we find similarly

$$u_j = C_{jk}\varphi_k, \qquad D_{jk}\varphi_k = ((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{jk}\varphi_k = 0; \quad (21)$$

$$\begin{aligned}\varphi_j = B^*_{ji}\tilde{u}_i = B_{ij}\tilde{u}_i = (\mu\partial_{pp} - \rho\partial_{..})\delta_{ij}\tilde{u}_i = (\mu\partial_{pp} - \rho\partial_{..})\tilde{u}_j, \quad A_{ij}\tilde{u}_j = 0,\\ u_j = C_{jk}B^*_{ki}\tilde{u}_i = C_{jk}B_{ik}\tilde{u}_i = C_{jk}(\mu\partial_{pp} - \rho\partial_{..})\delta_{ik}\tilde{u}_i = C_{jk}(\mu\partial_{pp} - \rho\partial_{..})\tilde{u}_k.\end{aligned} \quad (22)$$

In (21), φ_j satisfy the wave equation rather than the product of two wave operators, as in (17). But instead, the multiplier $(\mu\partial_{pp} - \rho\partial_{..})$ appears in (22).

For variant 3, we have

$$u_j = C_{jk}\varphi_k, \qquad D_{jk}\varphi_k = (\mu\partial_{pp} - \rho\partial_{..})\delta_{jk}\varphi_k = 0; \quad (23)$$

$$\begin{aligned}\varphi_j = B^*_{ji}\tilde{u}_i = B_{ij}\tilde{u}_i = ((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{ij}\tilde{u}_i = ((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\tilde{u}_j, \quad A_{ij}\tilde{u}_j = 0,\\ u_j = C_{jk}B^*_{ki}\tilde{u}_i = C_{jk}B_{ik}\tilde{u}_i = C_{jk}((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\delta_{ik}\tilde{u}_i = C_{jk}((\lambda+2\mu)\partial_{ss} - \rho\partial_{..})\tilde{u}_k.\end{aligned} \quad (24)$$

Formulas (23) and (24) are analogous to (21) and (22), and only the multipliers $(\mu\partial_{pp} - \rho\partial_{..})$ and $((\lambda + 2\mu)\partial_{ss} - \rho\partial_{..})$ change places.

Thus, for the Galerkin solution the above three variants of the formulas are possible. Since for solution (17) $B_{ij}f_j = \delta_{ij}f_j = f_i = 0$, the forms (7) and (8) coincide and solution (17) is general. Solutions (21) and (23) will become complete if they are written in the form (7), where B and D correspond to variants 2 or 3.

We write the Papkovich–Neuber solution [1, 3, 4] for an isotropic material in the case of statics as in [27]

$$\begin{aligned}u_j = C_{jk}\varphi_k = (1+2\mu_1)\varphi_j - x_1\partial_j\varphi_1 - x_2\partial_j\varphi_2 - x_3\partial_j\varphi_3 - \partial_j\varphi_4,\\ D_{jk}\varphi_k = (1+\mu_1)\partial_{pp}\varphi_j = 0, \qquad \mu_1 = \mu/(\lambda+\mu).\end{aligned} \quad (25)$$

In this case the appropriate operators take the form

$$\begin{aligned}A_{ij} = \partial_{ij} + \mu_1\delta_{ij}\partial_{ss} = A^*_{ji}, \quad C_{jk} = (1+2\mu_1)\delta_{jk} - x_k\partial_j, \quad C^*_{kj} = 2(1+\mu_1)\delta_{jk} + x_k\partial_j,\\ B_{ij} = (2\mu_1 - 1)\delta_{ij} - x_j\partial_i, \quad B^*_{ji} = 2\mu_1\delta_{ij} + x_j\partial_i, \quad D_{jk} = (1+\mu_1)\delta_{jk}\partial_{pp} = D^*_{kj}, \quad x_4 = 1, \quad \partial_4 = 0\end{aligned} \quad (26)$$

and the relations $AC = BD$ and $C^*A = DB^*$ hold. From (26), we obtain the expression of the function φ_j via the solution of the Lamé equations and the formula for obtaining new solutions (symmetry operator):

$$\begin{aligned}\varphi_j = B^*_{ji}\tilde{u}_i = 2\mu_1\tilde{u}_j + x_j\partial_i\tilde{u}_i, \quad \tilde{u}_4 = 0, \quad A_{ij}\tilde{u}_j = \partial_{ij}\tilde{u}_j + \mu_1\partial_{ss}\tilde{u}_i = 0,\\ u_j = C_{jk}B^*_{ki}\tilde{u}_i = \{2\mu_1[(1+2\mu_1)\delta_{ji} + x_j\partial_i - x_i\partial_j] - x_kx_k\partial_{ji}\}\tilde{u}_i\\ = 2\mu_1[(1+2\mu_1)\tilde{u}_j + x_j\partial_i\tilde{u}_i - x_i\partial_j\tilde{u}_i] - (x_1^2 + x_2^2 + x_3^2 + 1)\partial_{ji}\tilde{u}_i.\end{aligned} \quad (27)$$

It follows from (27) that functions φ_j are connected with each other by the relation

$$\partial_j\varphi_j = 2\mu_1\varphi_4 + \partial_j(x_j\varphi_4). \quad (28)$$

Let us show the generality of solution (25), i.e., check the condition $D\,\mathrm{Ker}\,C = \mathrm{Ker}\,B$. The kernel of the operator C is determined from the equations

$$C_{jk}g_k = [(1+2\mu_1)\delta_{jk} - x_k\partial_j]g_k = [2(1+\mu_1)\delta_{jk} - \partial_jx_k]g_k = 2(1+\mu_1)g_j - \partial_jx_kg_k = 0.$$

These equations will hold if g_j is taken in the form

$$g_j = \partial_jg, \qquad j = 1,2,3, \qquad g_4 = -x_ig_i + 2(1+\mu_1)g, \qquad i \leqslant 3 \quad (29)$$

(g is an arbitrary function), i.e., (29) is $\mathrm{Ker}\,C$.

The kernel of the operator B is found from the equations

$$B_{ij}f_j = [(2\mu_1 - 1)\delta_{ij} - x_j\partial_i]f_j = (2\mu_1\delta_{ij} - \partial_i x_j)f_j = 2\mu_1 f_i - \partial_i x_j f_j = 0,$$

which will always be fulfilled if we set

$$f_i = \partial_i f, \qquad i = 1,2,3, \qquad f_4 = -x_s f_s + 2\mu_1 f, \qquad s \leqslant 3 \tag{30}$$

(f is an arbitrary function), i.e., (30) is Ker$B$. It is evident that C is obtained from B by replacing the coefficient μ_1 by $1+\mu_1$; a similar replacement is also made for the kernels of (29) and (30).

Now we find $D\,\mathrm{Ker}\,C$:

$$\begin{gathered} D_{jk}g_k = (1+\mu_1)\partial_{pp}g_j, \qquad (1+\mu_1)\partial_{pp}g_j = (1+\mu_1)\partial_{pp}\partial_j g = \partial_j[(1+\mu_1)\partial_{pp}g], \qquad j = 1,2,3, \\ (1+\mu_1)\partial_{pp}g_4 = (1+\mu_1)\partial_{pp}[-x_i g_i + 2(1+\mu_1)g] = (1+\mu_1)(-x_i\partial_{pp}g_i + 2\mu_1\partial_{pp}g), \qquad i \leqslant 3. \end{gathered} \tag{31}$$

If in (31) we denote

$$f = (1+\mu_1)\partial_{pp}g, \qquad f_i = (1+\mu_1)\partial_{pp}g_i = \partial_i[(1+\mu_1)\partial_{pp}g] = \partial_i f, \qquad i = 1,2,3,$$

$$f_4 = (1+\mu_1)\partial_{pp}g_4 = -x_i(1+\mu_1)\partial_{pp}g_i + 2\mu_1(1+\mu_1)\partial_{pp}g = -x_i f_i + 2\mu_1 f, \qquad i \leqslant 3,$$

then (31) takes the form (30), i.e., we obtain that $D\,\mathrm{Ker}\,C = \mathrm{Ker}\,B$.

Thus, the Papkovich–Neuber solution is general and complete, i.e., the form (25) contains all solutions of Lame's equations. It follows from (27) and (28) that, in the general case, one cannot assume that the function φ_4 is equal to zero, as many authors (beginning from P. F. Papkovich) did [2, 3]. The values of Poisson's ratio $\nu = -3/4, -1/2, -1/4, 0$, and $1/4$ are not exceptional, but are associated with attempts in [3, 15, 16, 21] to prove the generality of the Papkovich–Neuber solution for the case $\varphi_4 = 0$.

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